Stone's Theorem

In this section we are going to use the functional calculus we developed in the Spectral Theorem theory to study operators of the form $U(t) = e^{itA}$ where A is a self-adjoint operator. We start describing some properties of these operators.

Theorem 1. *Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$ and let $U(t) = e^{itA}$. Then*

(i) $U(t)$ is a unitary operator for all $t \in \mathbb{R}$ and $U(t+s) = U(t)U(s)$ for all $s, t \in \mathbb{R}$. Furthermore, $\{U(t)\}_{t\in\mathbb{R}}$ forms an Abelian group under composition of operators.

(ii) $U(t)\varphi \to U(t_0)\varphi$ for all $\varphi \in \mathcal{H}$ as $t \to t_0$, i.e. $t \mapsto U(t)$ is a continuous with respect to the strong operator topology.

(iii) $\dfrac{U(t)\psi - \psi}{t} \to iA\psi$ *for all* $\psi \in D(A)$ *as* $t \to 0$*.*

(iv) If $\lim\limits_{t\to 0}\dfrac{U(t)\psi - \psi}{t}$ *exists, then* $\psi \in D(A)$*.*

Proof.

(i). It follows directly for the functional calculus and the properties of the complex value function $h_t(\lambda) = e^{it\lambda}$.

We shall write

$$U(t)^*U(t) = \Phi_A(h_t)^*\Phi_A(h_t) = \Phi_A(\overline{h_t})\Phi_A(h_t) = \Phi_A(\overline{h_t}h_t) = \Phi_A(1) = I.$$

Thus $U(t)$ is unitary.

In addition,

$$U(t)U(s) = \Phi_A(h_t)\Phi_A(h_s) = \Phi_A(h_t \cdot h_s) = \Phi_A(h_{t+s}) = U(t+s).$$

To show that $U(t)$ form a group we notice that we already have proved that it is closed under composition. The associativity and commutativity can be proved using the above procedure. We observe that $U(-t)$ is the inverse of $U(t)$ and $U(0)$ is the neutral element.

(ii). To prove this we first observe that it is enough to show that $t \mapsto U(t)$ is strongly continuous at $t = 0$.
It is convenient to use the projection-valued measure formulation. Then

$$\|e^{itA}\varphi - \varphi\|^2 = \int_{\mathbb{R}} |e^{it\lambda} - 1|^2 \, d\langle E^A_\lambda \varphi, \varphi\rangle$$

since for any function h

$$\begin{aligned}\|h(A)\varphi\|^2 &= \langle h(A)\varphi, h(A)\varphi\rangle = \langle \varphi, h(A)^* h(A)\varphi\rangle \\ &= \langle \varphi, \overline{h(A)} h(A)\varphi\rangle = \langle \varphi, (|h|^2)(A)\varphi\rangle.\end{aligned}$$

Since $|e^{it\lambda} - 1|^2$ is dominated by the integrable function $g(\lambda) = 4$, and $|e^{it\lambda} - 1|^2 \to 0$ pointwise for all λ as $t \to 0$, we have that

$$\|U(t)\varphi - \varphi\|^2 \to 0$$

by the Lebesgue dominated convergence theorem. Thus $t \mapsto U(t)$ is continuous at $t = 0$.

(iii). We can employ a similar technique to prove (iii). We see that

$$\left\|\frac{U(t)\psi - \psi}{t} - iA\psi\right\|^2 = \int_{\mathbb{R}} \left|\frac{e^{it\lambda} - 1}{t} - i\lambda\right|^2 d\langle E_\lambda^A \psi, \psi\rangle$$

On the other hand, we observe that

$$|e^{ix} - 1|^2 = 4\sin^2(\frac{x}{2}) \leq x^2.$$

Hence

$$\Big|\frac{e^{it\lambda} - 1}{t} - i\lambda\Big|^2 \leq \Big(\Big|\frac{\lambda t}{t}\Big| + |\lambda|\Big)^2 = (2\lambda)^2$$

which is integrable because of

$$\int_{\mathbb{R}} |2\lambda|^2 \, d\langle E_\lambda^A \psi, \psi\rangle = \|2A\psi\|^2 < \infty$$

as $\psi \in D(A)$. Since $\Big|\frac{e^{it\lambda}-1}{t} - i\lambda\Big|^2 \to 0$ pointwise for all $\lambda \in \mathbb{R}$ as $t \to 0$, the Lebesgue dominated convergence theorem yields the result.

(iv). We define

$$D(B) = \left\{\psi : \lim_{t\to 0}\frac{U(t)\psi - \psi}{t} \text{ exists}\right\}$$

and let

$$iB\psi = \lim_{t\to 0}\frac{U(t)\psi - \psi}{t}.$$

Then it is easy to show that B is symmetric, i.e. $B \subseteq B^*$. From (iii) we have that $A \subseteq B$. Since A is self-adjoint we have that $A = B$.

□

Definition 1. *Let* $\{U(t)\}_{t\in\mathbb{R}}$ *be a family of unitary operators such that* $U(t)U(s) = U(t+s)$ *for all* $t, s \in \mathbb{R}$*. If in addition it holds that* $U(t)\varphi \to U(t_0)\varphi$ *for all* $\varphi \in \mathcal{H}$ *as* $t \to t_0$*, we call* $\{U(t)\}_{t\in\mathbb{R}}$ *a strongly continuous (one-parameter) unitary group.*

Remark 1. *The Stone Theorem is essentially the converse of the Theorem 1. Combining Theorem 1 and Stone's Theorem it is established a bijection between strongly continuous one-parameter unitary groups and self-adjoint operators on a Hilbert space.*

Theorem 2 (Stone's Theorem). *Let* $\{U(t)\}_{t\in\mathbb{R}}$ *be a strongly continuous one-parameter unitary group on a Hilbert space* $\mathcal{H}$*. Then there exists a unique self-adjoint operator* A *on* $\mathcal{H}$ *such that* $U(t) = e^{itA}$*.*

Definition 2. *If* $\{U(t)\}_{t\in\mathbb{R}}$ *is a strongly continuous one-parameter unitary group, then the self-adjoint operator* A *with* $U(t) = e^{itA}$ *is called the infinitesimal generator of* $U(t)$*.*

Proof. We observe from (iii) in Theorem 1 that A can be obtained by differentiating $U(t)$ at $t = 0$. We will see that this can done on a dense subset of $\mathcal{H}$ consisting of suitable vectors. This will yield an operator which we will show to be essentially self-adjoint by using the basic criteria. We will see that $U(t)$ is the exponential of the closure of this operator.

First we consider $f \in C_0^\infty(\mathbb{R})$. For each $\varphi \in \mathcal{H}$ we define

$$\varphi_f = \int_{-\infty}^{\infty} f(t)U(t)\varphi \, dt. \tag{0.1}$$

This integral is Hilbert space-valued and defined as a Riemann integral, which is well-defined since $U(t)$ is strongly continuous.

Let D be the set of finite linear combinations of all such φ_f for $\varphi \in \mathcal{H}$ and $f \in C_0^\infty(\mathbb{R})$. We use the approximate identity $\phi_\epsilon(x) = \frac{1}{\epsilon}\phi(\frac{x}{\epsilon})$ where $\phi \in C^\infty(\mathbb{R})$ is a nonnegative function with support contained in $(-1,1)$ and $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$. Then the Minskowskii inequality in Banach spaces and the properties of ϕ_ϵ give us

$$\begin{aligned} \|\varphi_{\phi_\epsilon} - \varphi\| &= \left\| \int_{-\infty}^{\infty} \phi_\epsilon(t)(U(t)\varphi - \varphi)\,dt \right\| \\ &\leq \int_{-\infty}^{\infty} \phi_\epsilon(t)\,\|U(t)\varphi - \varphi\|\,dt \\ &\leq \int_{-\infty}^{\infty} \phi_\epsilon(t)\,dt \sup_{t\in[-\epsilon,\epsilon]} \|U(t)\varphi - \varphi\| \\ &= \sup_{t\in[-\epsilon,\epsilon]} \|U(t)\varphi - \varphi\|. \end{aligned} \tag{0.2}$$

We conclude that D is dense in $\mathcal{H}$ since letting ϵ tending to zero, we have $\varphi_{\phi_\epsilon} \to \varphi$ because $U(t)$ is strongly continuous.

For a $\varphi_f \in D$

$$
\begin{aligned}
\Big(\frac{U(s)-I}{s}\Big)\varphi_f &= \Big(\frac{U(s)-I}{s}\Big)\int_{-\infty}^{\infty} f(t)\,U(t)\varphi\,dt \\
&= \int_{-\infty}^{\infty} f(t)\,\Big(\frac{U(s+t)-U(t)}{s}\Big)\varphi\,dt \\
&= \int_{-\infty}^{\infty} f(t)\,\Big(\frac{U(s+t)}{s}\Big)\varphi\,dt - \int_{-\infty}^{\infty} f(t)\,\Big(\frac{U(t)}{s}\Big)\varphi\,dt \\
&= \int_{-\infty}^{\infty} f(t-s)\,\Big(\frac{U(t)}{s}\Big)\varphi\,dt - \int_{-\infty}^{\infty} f(t)\,\Big(\frac{U(t)}{s}\Big)\varphi\,dt \\
&= \int_{-\infty}^{\infty} \frac{f(t-s)-f(t)}{s}\,U(t)\varphi\,dt.
\end{aligned}
$$

If we let $s \to 0$ the last term converges to

$$-\int_{-\infty}^{\infty} f'(t)\, U(t)\varphi\, dt = \varphi_{-f'}$$

since $\dfrac{f(t-s)-f(t)}{s}$ converges uniformy to $-f'(t)$ as $s \to 0$.
Now we can define the operator $\widetilde{A}$ on D by $\widetilde{A}\varphi_f = -i\varphi_{-f'}$. By definition $\widetilde{A} : D \to D$. Observe that $U(t) : D \to D$. Indeed,

$$\begin{aligned} U(s)\varphi_f = U(s) \int_{-\infty}^{\infty} f(t)\, U(t)\varphi\, dt &= \int_{-\infty}^{\infty} f(t)\, U(s+t)\varphi\, dt \\ = \int_{-\infty}^{\infty} g(t)\, U(t)\varphi\, dt = \varphi_g \end{aligned}$$

where $g(t) = f(t-s)$. Moreover $U(t)\widetilde{A}\varphi_f = \widetilde{A}U(t)\varphi_f$ for $\varphi_f \in D$,

$$\widetilde{A}U(s)\varphi_f = \widetilde{A}\varphi_g = -i\varphi_{-g'} = -iU(s)\varphi_{-f'} = U(s)\widetilde{A}\varphi_f, \qquad (0.3)$$

where we again use $g(t) = f(t-s)$. The identity (0.3) tells us that differentiation and translation commute.

Next we show that $\widetilde{A}$ is symmetric. We write

$$\begin{aligned}\langle \widetilde{A}\varphi_f, \varphi_g\rangle &= \lim_{s\to 0}\Big\langle \Big(\frac{U(s)-I}{is}\Big)\varphi_f, \varphi_g\Big\rangle = \lim_{s\to 0}\Big\langle \varphi_f, \Big(\frac{I-U(-s)}{is}\Big)\varphi_g\Big\rangle \\ &= \langle \varphi_f, -i\varphi_{-g'}\rangle = \langle \varphi_f, \widetilde{A}\varphi_g\rangle.\end{aligned}$$

Using the basic criteria we proceed to prove that $\widetilde{A}$ is essentially self-adjoint. Suppose there is a $\psi \in D(\widetilde{A}^*)$ such that $\widetilde{A}^*\psi = i\psi$. Then for each $\varphi \in D(\widetilde{A}) = D$ we have

$$\begin{aligned}\frac{d}{dt}\langle U(t)\varphi, \psi\rangle &= \lim_{s\to 0}\Big\langle\Big(\frac{U(t+s)-U(t)}{s}\Big)\varphi, \psi\Big\rangle \\ &= \lim_{s\to 0}\Big\langle\Big(\frac{U(s)-I}{s}\Big)U(t)\varphi, \psi\Big\rangle \\ &= \langle i\widetilde{A}U(t)\varphi, \psi\rangle = i\langle U(t)\varphi, \widetilde{A}^*\psi\rangle \\ &= i\langle U(t)\varphi, i\psi\rangle = \langle U(t)\varphi, \psi\rangle.\end{aligned}$$

Hence the complex value function $f(t) = \langle U(t)\varphi, \psi\rangle$ satisfies the ordinary differential equation $f' = f$, demanding an exponential solution $f(t) = f(0)\, e^t$. On the other hand, $U(t)$ is unitary and thus has norm 1.

Thus $f(t)$ has to be bounded for positive and negative t, which is only possible if $f(0) = 0 = \langle \varphi, \psi\rangle$.

Since D is dense in $\mathcal{H}$ and φ was chosen arbitrarily, we conclude that $\psi = 0$.

Similarly, we conclude that the equation $\widetilde{A}^*\psi = -i\psi$ has no nonzero solutions. Then by the basic criteria for essentially self-adjointness it follows that $\widetilde{A}$ is essentially self-adjoint on D, that is, $A = \overline{\widetilde{A}}$ is self-adjoint.

We then define $V(t) = e^{itA}$ and prove that $U(t)$ and $V(t)$ coincide on D. Let $\varphi \in D$. Since $\varphi \in D(A)$, $V(t)\varphi \in D(A)$ and

$$V'(t)\varphi = iAV(t)\varphi$$

by (iii) in Theorem 1.

We already know that $U(t)\varphi \in D \subseteq D(A)$ for all $t \in \mathbb{R}$. If we set

$$w(t) = U(t)\varphi - V(t)\varphi,$$

then $w(t)$ is a differentiable Hilbert space-valued function, since $U(t)$ is strongly differentiable by assumption and $V(t)$ because of Theorem 1. We obtain

$$w'(t) = i\widetilde{A}U(t)\varphi - iAV(t)\varphi = iAw(t).$$

Hence,

$$\frac{d}{dt}\|w(t)\|^2 = -i\langle Aw(t), w(t)\rangle + i\langle w(t), Aw(t)\rangle = 0$$

which implies that $w(t) = 0$ for all $t \in \mathbb{R}$ since $w(0) = 0$ by definition.

This means that $U(t)\varphi = V(t)\varphi$ for all $t \in \mathbb{R}$. Thus we have found A to be the infinitesimal generator of $U(t)$.

Finally, we prove the uniqueness. Suppose that there exists a self-adjoint operator B such that $e^{itB} = U(t) = e^{itA}$. Then by (iii) in Theorem 1 $A = B$. $\square$

Applications

In this part we will discuss some applications of the Stone Theorem.

Definition 3. *For $\varphi \in L^2(\mathbb{R})$ we define the* **translation** *operator by*

$$(U(a)\varphi)(x) = \varphi(x+a).$$

i.e. $U(a)$ shifts the function $\varphi(x)$ to the left by a.

From the definition we can see that $U(a)$ is an isometry since the Lebesgue integral is translation-invariant and since translations are invertible, we have a unitary map for all $a \in \mathbb{R}$. Since

$$(U(a)U(b)\varphi)(x) = (U(b)\varphi)(x+a) = \varphi(x+a+b) = (U(a+b)\varphi)(x)$$

we can conclude that $\{U(a)\}_{a\in\mathbb{R}}$ forms a one-parameter group.

The goal of the next exercises is proving that the translation operator group is also strongly continuous and thus we have a strongly continuous one-parameter unitary group.

Exercise 3. *Let X be a Banach space and let $\mathcal{T} \subset \mathcal{B}(X)$ be bounded, i.e.* $\sup_{T\in\mathcal{T}}\|T\| = c < \infty$. *Then in $\mathcal{T}$ are equivalent:*

(i) strong convergence,

(ii) strong convergence on a dense subspace M of X.

Exercise 4. *The translation group on $L^p(\mathbb{R}^n)$ $(1 \leq p < \infty)$ is strongly continuous, i.e. if we define*

$$(\tau_{\overrightarrow{a}} f)(\overrightarrow{x}) = f(\overrightarrow{x} + \overrightarrow{a}),$$

then

$$\lim_{|\overrightarrow{a}|\to 0} \tau_{\overrightarrow{a}} f = f$$

in $L^p(\mathbb{R}^n)$ for all $f \in L^p(\mathbb{R}^n)$.

Now we have that the translation forms a strongly continuous one-parameter unitary group. By Stone's Theorem there exists a self-adjoint infinitesimal generator A, such that $U(t) = e^{itA}$. From Theorem 1 (iii) and (iv) we know that $D(A)$ is given by all functions in $\psi \in L^2(\mathbb{R})$ with

$$\lim_{t\to 0} \frac{U(t)\psi - \psi}{t} = \lim_{t\to 0} \frac{\psi(\cdot + t) - \psi(\cdot)}{t} \quad \text{exists},$$

and therefore

$$A\psi = -i \lim_{t\to 0} \frac{U(t)\psi - \psi}{t}.$$

If ψ were a differentiable function the pointwise limit above will give us the product of $-i$ times the derivative. However, we have to consider the limit in L^2. Then we need to prove that there exists a function $\varphi \in L^2(\mathbb{R})$ so that

$$\lim_{t\to 0}\left\|\frac{U(t)\psi - \psi}{t} - \varphi\right\|^2 = \lim_{t\to 0}\int_{\mathbb{R}}\left|\frac{\psi(s+t)-\psi(s)}{t} - \varphi(s)\right|^2 ds = 0.$$

To prove this we use weak derivatives, that is, if $\lim_{t\to 0}\dfrac{U(t)\psi - \psi}{t} = \varphi$ exists, then $\langle \varphi, \eta\rangle = -\langle \varphi, \eta'\rangle$, for all $\eta \in C_0^\infty(\mathbb{R})$. Indeed, Let $\eta \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}\langle \varphi, \eta \rangle &= \left\langle \lim_{t\to 0} \frac{U(t)\psi - \psi}{t}, \eta \right\rangle = \lim_{t\to 0}\left\langle \frac{U(t)\psi - \psi}{t}, \eta \right\rangle \\ &= \lim_{t\to 0}\left\langle \psi, \frac{U(-t)\eta - \eta}{t} \right\rangle = \left\langle \psi, \lim_{t\to 0} \frac{U(-t)\eta - \eta}{t} \right\rangle \\ &= -\left\langle \psi, \lim_{t\to 0} \frac{U(t)\eta - \eta}{t} \right\rangle = -\langle \varphi, \eta' \rangle\end{aligned}$$

where η' is usual derivative of η (as a pointwise limit). It is clear that η' is also the L^2-limit of $\lim_{t\to 0} \frac{U(t)\psi - \psi}{t}$ as $t \to 0$ (exercise).

In conclusion we have that the domain of A can be formally written as

$$D(A) = \left\{ \psi \in L^2(\mathbb{R}) : \lim_{t \to 0} \frac{U(t)\psi - \psi}{t} \text{ exists and is in } L^2(\mathbb{R}) \right\}$$

Setting $D(D) = D(A)$ we define the operator D as the map taking a function in its weak derivative. By Theorem 1

$$A = -iD,$$

and thus

$$U(t) = e^{tD},$$

which formally written as a power series corresponds to Taylor's theorem.

Exercise 5. *Consider the linear initial value problem*

$$\begin{cases} \partial_t u(x,t) = i\Delta u(x,t), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x). \end{cases} \tag{0.4}$$

(i) Show that solutions of (0.4) *form a strongly continuous one-parameter unitary group in* $L^2(\mathbb{R}^n)$.

(ii) Prove that the infinitesimal generator operator A *in* (0.4) *is the Laplacian* Δ *with* $D(A) = H^2(\mathbb{R}^n)$.

Example 1. *Let* $X = L^2(\mathbb{R})$ *and* $Y = H^s(\mathbb{R})$ *with* $s \geq 3$. *We define the operator* A_0 *by* $D(A_0) = H^3(\mathbb{R})$ *and* $A_0 u = D^3 u$ *for* $u \in D(A_0)$ *where* $D = \dfrac{d}{dx}$.

A_0 *is the infinitesimal generator of a* C_0 *group of isometries on* X.

To see this we prove that A_0 *is a skew-adjoint operator, i.e.* iA_0 *is self-adjoint or equivalently* $(A_0 u, u) = 0$ *for all* $u \in D(A_0)$. *This follows easily from*

$$(A_0 u, u) = \int D^3 u \cdot u \, dx = -\int u \cdot D^3 u \, dx = -(A_0 u, u)$$

where we have integrated by parts three times. From Stone's theorem it follows that A_0 *is the infinitesimal generator of a unitary group on* $X = L^2(\mathbb{R})$.

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